9.4.1 ROTATIONAL MOTION[M34]

9.4.1.1 Rotational Quantities

When an object spins, it is said to undergo **rotational motion**. The **axis of rotation** is the line about which the rotation occurs. A point on an object that rotates about a single axis undergoes **circular motion** around that axis—*i.e.* the point travels in a circle around the axis of rotation.

In science, we often measure angles in radians, rather than degrees. A **radian** (rad) is an angle whose arc length is equal to its radius (~57.3°). Thus, there are 2π radians in a circle:

$$2\pi \text{ rad} = 360°$$

The radian is a pure number, with no dimensions, since it is the ratio of an arc length (a distance) to the length of a radius (also a distance).

9.4.1.1.1 Angular Displacement

Angular displacement describes how much an object has rotated:

$$\text{angular displacement (in radians)} = \frac{\text{change in arc length}}{\text{distance from axis}}$$

$$\Delta\theta = \frac{\Delta s}{r}$$

9.4.1.1.2 Angular Speed

Angular speed describes how quickly rotation occurs, or the rate of rotation:

$$\text{average angular speed} = \frac{\text{angular displacement}}{\text{time interval}}$$

$$\omega_{avg} = \frac{\Delta\theta}{\Delta t}$$

9.4.1.1.3 Angular Acceleration

Angular acceleration occurs when angular speed changes:

$$\text{average angular acceleration} = \frac{\text{change in angular speed}}{\text{time interval}}$$

$$\alpha_{\text{avg}} = \frac{\omega_2 - \omega_1}{t_2 - t_1} = \frac{\Delta\omega}{\Delta t}$$

9.4.1.1.4 Equations of Motion

Rotational and Linear Kinematic Equations

Rotational motion with constant angular acceleration	Linear motion with constant acceleration
$\omega_f = \omega_i + \alpha\Delta t$	$v_f = v_i + a\Delta t$
$\Delta\theta = \omega_i\Delta t + \frac{1}{2}\alpha(\Delta t)^2$	$\Delta x = v_i\Delta t + \frac{1}{2}a(\Delta t)^2$
$\omega_f^2 = \omega_i^2 + 2\alpha(\Delta\theta)$	$v_f^2 = v_i^2 + 2a(\Delta x)$
$\Delta\theta = \frac{1}{2}(\omega_i + \omega_f)\Delta t$	$\Delta x = \frac{1}{2}(v_i + v_f)\Delta t$

9.4.1.1.5 Tangential Speed

The tangential speed of an object rotating about an axis is simply the instantaneous linear speed of that object, directed along the tangent to the circular path of the object.

$$\text{tangential speed} = \text{distance from axis} \times \text{angular speed}$$

$$v_t = r\theta$$

9.4.1.1.6 Tangential Acceleration

Tangential acceleration is the instantaneous linear acceleration of an object directed along the tangent to the object's circular path.

$$\text{tangential acceleration} = \text{distance from axis} \times \text{angular acceleration}$$

$$a_t = r\alpha$$

9.4.1.1.7 Centripetal Acceleration

Centripetal acceleration is the acceleration directed towards the centre of a circular path.

$$\text{centripetal acceleration} = \frac{(\text{tangential speed})^2}{\text{distance from axis}}$$

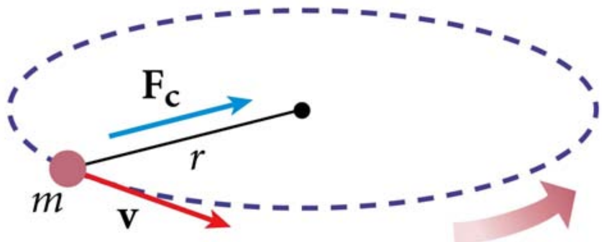
$$a_c = \frac{v_t^{\,2}}{r}$$

$$\text{centripetal acceleration} = \text{distance from axis} \times (\text{angular speed})^2$$

$$a_c = r\omega^2$$

9.4.1.1.8 The Force that Maintains Circular Motion

The net force on an object directed towards the centre of the object's circular path is the force that maintains the object's circular motion.



$$\text{force that maintains circular motion} = \text{mass} \times \frac{(\text{tangential speed})^2}{\text{distance from axis}}$$

$$F_c = \frac{mv_t^{\,2}}{r}$$

$$\text{force that maintains circular motion} = \text{mass} \times \text{distance from axis} \times (\text{angular speed})^2$$

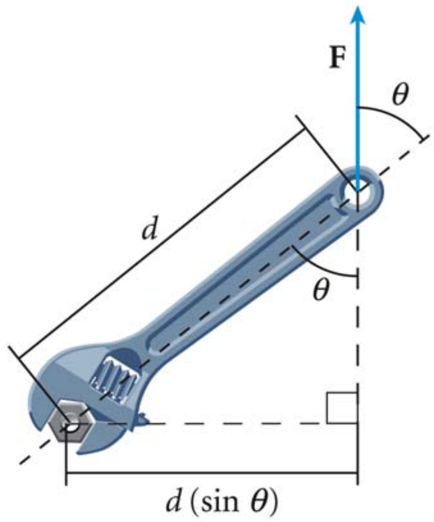
$$F_c = mr\omega^2$$

9.4.1.2 Torque

Torque is a measure of the ability of a force to rotate an object around some axis. How easily an object rotates depends not only on how much force is applied but also on where the force is applied. The farther the force is from the axis of rotation, the more torque is produced, and the easier it is to rotate the object. The perpendicular distance from the axis of rotation to a line drawn along the direction of the force is called the lever arm, or moment arm.

The magnitude of a torque is also dependent on the angle between the force and the lever arm.

$$\text{torque} = \text{force} \times \text{lever arm}$$
$$\tau = Fd(\sin\theta)$$



The illustration shows a wrench pivoted around a bolt.. In this case, the applied force acts at an angle to the wrench. The quantity *d* is the distance from the axis of rotation to the point where force is applied. The quantity $d(\sin\theta)$, however, is the perpendicular distance from the axis of rotation to a line drawn along the direction of the force, so it is the lever arm.

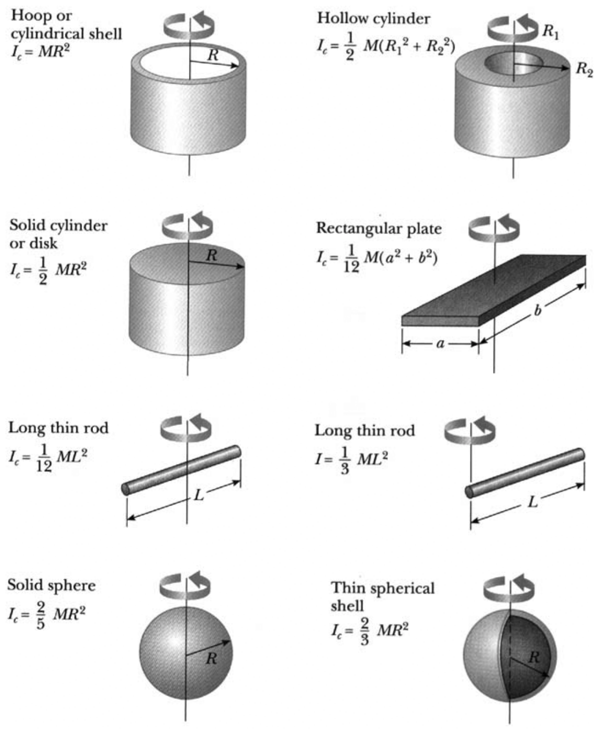
The SI unit for torque is the N•m.

9.4.1.3 Inertia

9.4.1.3.1 Moment of Inertia

Moment of inertia, which resists changes in rotational motion, is the rotational analogue of mass, which resists changes in translational motion There is, however, a fundamental difference between the two: mass is an intrinsic property of an object, but the moment of inertia is not. The moment of inertia depends both on the mass of an object, and the distribution of its mass about its axis of rotation. The further the mass of an object is distributed from its axis of rotation, the greater its moment of inertia and the more difficult it is to rotate.

The units for moment of inertia are kg•m^2.



9.4.1.3.2 Rotational Equilibrium

If the net force on an object is zero, the object is in *translational equilibrium*. If the net torque on an object is zero, the object is in *rotational equilibrium*. For an object to be completely in equilibrium, it must be in both translational and rotational equilibrium: both the net force and the net torque must be zero.

9.4.1.3.3 Newton's Second Law for Rotation

Just as Newton's Second Law can be expressed by the equation:

$$F_{net} = ma$$

when considering linear motion, for rotating motion it can be expressed as:

net torque = moment of inertia × angular acceleration

$$T_{net} = I\alpha$$

9.4.1.4 Angular Momentum

For rotary motion, the relationship between impulse and the change of angular momentum is similar to that for linear motion. Using the symbols for rotary motion, the equation becomes

$$T_t = I\omega_f - I\omega_i$$

where T_t is the angular impulse; and
$I\omega_f - I\omega_i$ is the change in angular momentum.

The dimensions of both angular impulse and angular momentum are kg·m^2/s. Just as the linear momentum of an object is unchanged unless a net external force acts on it, the angular momentum of an object is unchanged unless a net external torque acts on if. This is a statement of the law of conservation of angular momentum.

A rotating flywheel, which helps maintain a constant angular velocity of the crankshaft of an automobile engine, is an illustration—the rotational inertia of a fly-wheel is large.

Consequently torques acting on it do not produce rapid changes in its angular momentum. As the torque produced by the combustion in each cylinder tends to accelerate the crankshaft, the rotational inertia of the flywheel resists this action.

Similarly, as the torques produced in the cylinders where compression is occurring tend to decelerate the crankshaft, the rotational inertia of the flywheel resists this action and the flywheel tends to maintain a uniform rate of crankshaft rotation.

If the distribution of mass of a rotating object is changed, its angular velocity changes so that the angular momentum remains constant. A skater spinning on the ice with arms folded turns with relatively constant angular velocity. If she extends her arms, her rotational inertia increases. Since angular momentum is conserved, her angular velocity must decrease.

9.4.1.5 Kinetic Energy in Rotational Motion

Rotating objects possess kinetic energy, even though they may not be changing position.

When only the net force and torque are considered in these equations, all of the work done to produce rotation appears as kinetic energy. Since for rotational motion:

$$W = T\Delta\theta$$

the equation for kinetic energy is:

$$E = \tfrac{1}{2} I\omega^2$$

The wheel of a moving car has both linear motion and rotational motion. The wheel turns on its axle as the axle moves along parallel to the road. The kinetic energy of such an object is the sum of the kinetic energy due to linear motion and the kinetic energy due to rotary motion:

$$E = \tfrac{1}{2} mv^2 + \tfrac{1}{2} I\omega^2$$

9.4.1.6 Work in Rotational Motion

To compute the work done in rotary motion, we use the formula:

$$W = Fr\Delta\theta$$

Furthermore, torque is the product of a force and the length of its torque arm. In Figure 9.4.1.1, the torque arm is the radius of the circle, so:

$$T = Fr$$

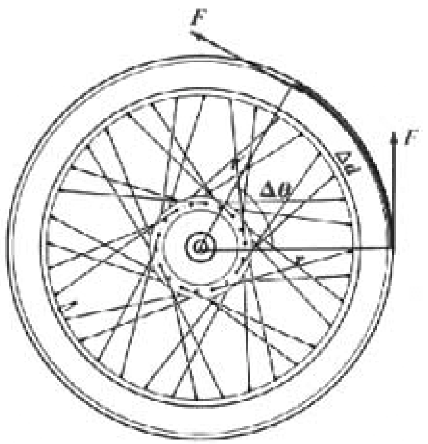


Figure 9.4.1.1
The work done on the wheel is equal to the product of the applied force, F, the radius of the wheel, r, and the displacement of the rim, $\Delta\theta$.

The work equation can then be written:

$$W = T\Delta\theta$$

which means that the work done in rotary motion can be computed by finding the product of the torque producing the motion and the angular displacement in radians.

9.4.1.7 Power in Rotational Motion

Power in rotational motion can be computed in the same way as work. Substituting the expression for work in rotational motion, $T\Delta\theta$, the power equation becomes:

$$P = \frac{T\Delta\theta}{\Delta t}$$

The time rate of angular displacement, $\Delta\theta/\Delta t$, is called the angular velocity, ω. For rotational motion, the expression for power then becomes:

$$P = T\omega$$

Which means that the power required to maintain rotational motion against an opposing torque is equal to the product of the torque maintaining the rotary motion and the constant angular velocity.

9.4.1.8 Spinning Objects and Gyroscopes

The **axis** of a **spinning object** maintains its direction. To change this direction a force is required.

The tendency for a spinning object to maintain its direction of spin accounts for

- the constant tilt of the earth's axis
- the use of a gyroscope as a compass

References

Holt Physics, Serway, R.A. and Faughn, J.S. (Holt, Rinehart and Winston, 2000) [ISBN 0-03-056544-8] Ch. 7-8

Work directly from text, with exercises:

7 Rotational Motion and the Law of Gravity

7.1 Measuring rotational motion

7.2 Tangential and centripetal acceleration

7.3 Causes of circular motion

8 Rotational Equilibrium and Dynamics

8.1 Torque

8.2 Rotation and inertia

8.3 Rotational dynamics